Title: Projectile Motion Analysis Using Parametric Equations

Brief Overview:

This unit integrates the subjects of trigonometry and physics to develop a model of projectile motion using parametric equations. Given a set of data provided in the lesson, the student will generate the parametric equations for x and y that models the movement of a projectile. The TI-82/83 calculator will be used to facilitate the model development.

Links to NCTM Standards:

- **Mathematics as Problem Solving**
  Students will apply the process of mathematical modeling to real- world projectile motion applications.

- **Mathematics as Communication**
  Students will examine and discuss the differences and similarities among graphs of projectile motion.

- **Mathematical Connections**
  Students will formulate mathematical links between projectile motion and mathematics.

- **Algebra**
  Students will represent variables in projectile motion using equations and graphs.

- **Functions**
  Students will analyze relationships found with projectile motion using tables, graphs, and equations.

- **Trigonometry**
  Students will apply right triangle trigonometry and analyze projectile motion models.

- **Statistics**
  Students will draw inferences from tables and graphs from real-world motion and use curve fitting to determine mathematical relationships.

- **Conceptual Underpinnings of Calculus**
  Students will analyze the polynomial graphs related to projectile motion.

- **Mathematical Structure**
  Students will investigate projectile motion relationships with functional and parametric systems.

Links to Maryland High School Core Learning Goals

Mathematics Core Learning Goals

- **1.1.1**
  The student will recognize, describe, and extend patterns and functional relationships that are expressed numerically, algebraically, and geometrically.

- **1.1.2**
  The student will represent patterns and functional relationships in a table, as a graph, and/or by a mathematical expression.

- **1.1.4**
  The student will describe the graph of a non-linear function in terms of the basic concepts of maxima and minima, roots, limits, rates of change, and continuity.

- **1.2.1**
  The student will determine the equation for a line, solve linear equations, and describe the solutions using numbers, symbols, and graphs.

- **1.2.4**
  The student will describe how the graphical model of a non-linear function represents a given problem and will estimate the solution.

- **1.2.5**
  The student will apply formulas and use matrices to solve real-world problems.

- **2.2.2**
  The student will solve problems in two-dimensional figures and/or right triangle geometry.

- **3.1.1**
  The student will design and/or conduct an investigation that uses statistical methods to analyze data and communicate results.

- **3.2.2**
  The student will make predictions by finding and using the line of best fit and by using a given curve of best fit.

Physics Core Learning Goals

- **5.1.1**
  The student will use analytical techniques appropriate to the study of physics.

- **5.1.2**
  The student will use algebraic and geometric concepts to describe an object's motion.

- **5.6.4**
  The student will use tables, graphs, and charts to display data in making arguments and claims in both written ands oral communications.

- **5.7.2**
  The student will recognize the important role that mathematics serves when solving problems in physics.

Grade/Level:

This activity can be used with Trigonometry or Physics in Grades 10-12.

Duration/Length:

Two to three class periods (variable)

Prerequisite Knowledge:

Students should have working knowledge of the following skills :

- Solving right triangles using trigonometry
- Using the graphing calculators to graph equations, data, and determine regression equations for linear and quadratic functions in function mode
- Using graphing calculator with parametric equations

Objectives:

Students will be able to:

- draw graphs of projectile motions with sample motions of vertical, horizontal, and at other angles of elevation.
- write the parametric equations for both horizontal and vertical components of distance for projectile motions.
- determine the equations to describe projectile motion using the graphing calculator.
- investigate the idea of mathematical modeling

Materials/Resources/Printed Materials:

- TI-82/83 Calculator
- Provided data and worksheets

Development/Procedures:

The concept of projectile motion of a baseball thrown by Cal Ripken in two dimensions is developed using the graphing calculator analyzing linear motion and accelerated motion in the function mode. After analysis in the function mode, the motion is examined in the parametric mode to find a more realistic representation of projectile motion.

The following assumptions were made to simplify the mathematical model:

Air resistance in the baseball's motion is negligibly small.
The vertical and the horizontal motions are independent since they are perpendicular.
The only external force on the baseball is caused by gravity.

Student Activity #1:
The students will examine the cases of a baseball thrown with a constant velocity of 100 ft/sec in a straight line, the baseball dropped from rest to the ground, and the baseball thrown at 100 ft/sec from the ground straight upward. They will analyze the data given in the tables to visualize the graph of the motion with lists on the calculator. In addition, the students will determine the regression equations for the three cases.

Student Activity #2:
The student will examine the case where Cal throws the baseball from third base to first base at 88 ft/ sec . This will be modeled by placing the horizontal equation (constant velocity) and the vertical equation (ball thrown upward) developed in Activity #1 into the x(t) and y(t) slots in the <y=> menu with the calculator in parametric mode. Then the model will investigate the case where Cal throws the baseball with an elevation (incident angle) between 0^{o} and 90^{o}. The parametric equations developed will be used to analyze the more general case of a baseball motion.

Time Limitations: One 90-minute or two 45-minute class periods are needed to complete the unit. The time depends upon the expertise that the students have with the graphing calculator.

Extension/Follow Up:

- Write the parametric equations developed in Activity #2 in a paper. In order to investigate the effect of certain variables upon projectile motion, the variables of initial velocity, angle of elevation, and gravitational acceleration could be varied and the graph of the motion examined. How would one design an exercise to examine these variations? Predict how these variables affect the parabola. Check your predictions. Explain why these changes occurred.
- Take the parametric equations developed for projectile motion with an incident angle between 0^o and 90^o and solve for y in terms of x. Graph the equation determined in the function mode on the graphing calculator using the initial velocity of 64 m/sec and an elevation angle of 30^o. Compare to the original graphs done in the parametric mode in Activity #2. What do you conclude? Why? Change these variables to other values and observe the changes in the graph. Explain the results.

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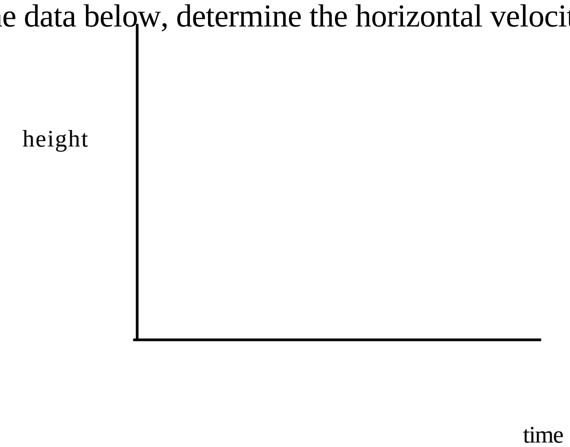
PROJECTILE MOTION: PART 1

When a player throws a ball into the air, it does not stay up forever. There are forces that act on the ball that cause it to follow a particular path. This exercise is to develop the mathematical equations that determine an object's path through the air.

I. CONSTANT VELOCITY

The velocity of any object traveling in a horizontal line can be determined from its distance traveled and the time elapsed. Using the data below, determine the horizontal velocity.

Time t (secs)	Distance x (ft)
0	0
0.5	50
1	100
1.5	150
2	200
2.5	250
3	300
3.5	350



A. Use your **calculator** to **graph** the data. Equation ________

1. Press S**tats** and E**dit**
2. Enter time in $\mathbf{L_1}$ and distance in $\mathbf{L_2}$
3. Press **2nd StatPlot**
4. Highlight **1:** and press E**nter**
5. Highlight **ON** press **Enter**
6. Put cursor on scatter plot press **Enter**
7. Set **Xlist** to $\mathbf{L_1}$
8. Set **Ylist** to $\mathbf{L_2}$
9. Press **ZOOM 9**

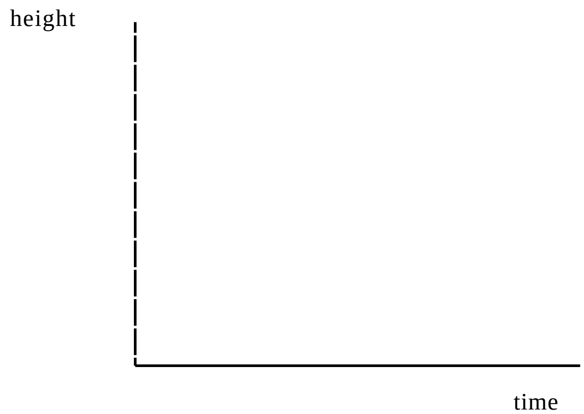
B. Determine regression equation.

10. Press **STAT**
11. Select **CALC**
12. Select the proper regression model.
13. Press **Y=**
14. Clear all y= or turn other y= off
15. Press **VARS** , then **5** to enter statistics mode.
16. Move cursor to **EQ** menu and press **7** (RegEQ)
17. Press **Enter**
18. Accurately sketch and label the graph and write the equation on the page above.

II. A FREE-FALLING OBJECT

Assume a ball is dropped from a height of 500 feet with no initial velocity. Use the following data to determine the acceleration due to gravity.

Timet(secs)	Heighty(ft.)
0	500
0.5	496
1	484
1.5	464
2	436
2.5	400
3	356
3.5	304
4	244
4.5	176
5	100
5.5	16



Equation:_______________________

A. Use your **calculator** to **graph** the data.

A. Use your calculator to graph the data.
1. Clear the data from part I
2. Press **Stats** then **4**
3. Press 2nd $\mathbf{L_1, L_2}$
4. Enter the data following the same procedure shown in part I.
5. Plot the points and observe the graph.

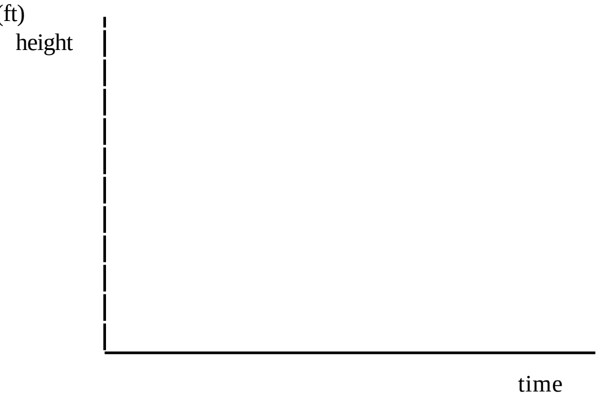
B. Determine a regression equation.

6. Follow the same procedure in Part I.
7. Select the correct regression form
8. Accurately sketch and label the graph and write the equation on the page above.

III. AN OBJECT THROWN STRAIGHT UP

A ballplayer throws a ball straight up in the air. Unless gravity has been repealed, it will come straight down after reaching some maximum height. Use the data from the table below to determine an equation for height.

Time t (secs)	Height y (ft)
0	0
0.5	46
1	84
1.5	114
2	136
2.5	150
3	156
3.5	154
4	144
4.5	126
5	100
5.5	66
6	24



equation ________________

A. Use your calculator to graph the data.

1. Clear the data from part I
2. Press **Stats** then **4**
3. Press 2nd $\mathbf{L_1, L_2}$
4. Enter the data following the same procedure shown in part I.
5. Plot the points and observe the graph.

B. Determine a regression equation.

6. Follow the same procedure in Part I.
7. Select the correct regression form
8. Neatly sketch and label the graph and write the equation on the page above.

IV. PRACTICE PROBLEMS

1. Determine the velocity of the ball in part I.

2. Write the velocity function in the horizontal direction (based on x).

3. Write position (x) in terms of velocity and time.

4. What would the position function be if the initial velocity was 85 ft/s?

5. How is the free falling object different from the object in part I?

6. Write the position (y) function for an object falling?

7. How do the graphs in part II and part III differ?

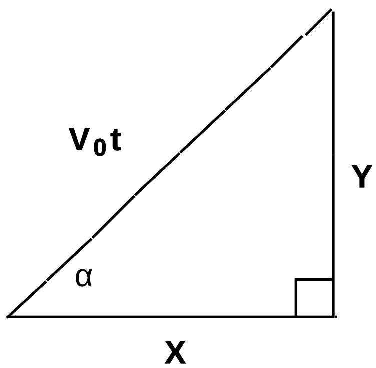
8. What accounts for this difference?

PROJECTILE MOTION: PART 2
'USING ALL THE ANGLES"

Most of the time balls are not thrown straight up or dropped straight down. Balls move at various angles of elevation off the ground. Trigonometry and physics can be used together to determine the path of objects thrown. There are two components of the velocity, horizontal and vertical. They are each independent of the other.

I. PARAMETRIC EQUATIONS.

To follow the path of a thrown ball we need to know how far horizontally and how high vertically the ball has traveled at each instant of time. The distance in each direction is determined by the way in which the ball was thrown.



Use trigonometry to solve for x in terms of the angle and the hypotenuse (v_ot).

Solve for y in the same way.

Correct the y function for the effect of gravity .(See the results of an object thrown straight up).

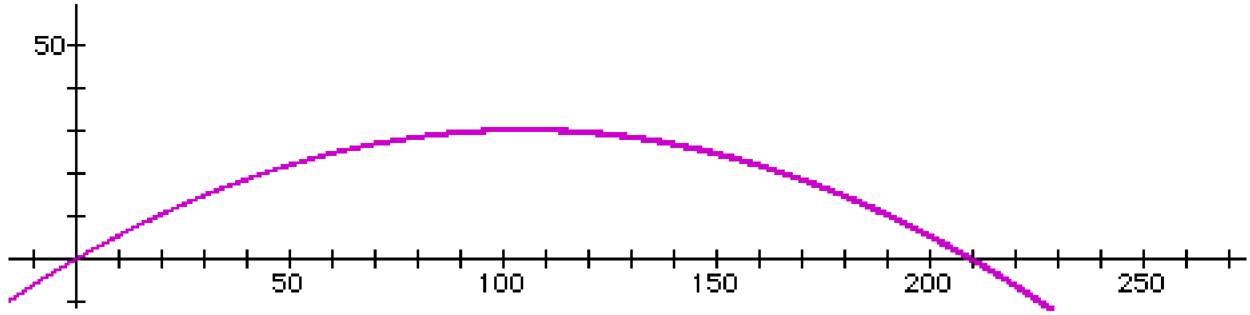
Parametric equations are equations in which the variables **x** any **y** are dependent upon a third variable **t** (time)**.**

Write the two parametric equations.

x=

y=

I. The graph below represents a famous Cal Ripken throw with an initial velocity of 88 ft/s at an angle of 30^{o}.



Graph your equation, using parametric mode, and compare your results to the graph above. Use the **Trace** function on your calculator to see if the two graphs compare. If yours is different, look back at your equations and change them. Keep trying until you get the correct graph.

Determine the distance from third base to first. Check the height of the ball at that distance. (From first to home is 90 ft.).
How good was Cal's throw?

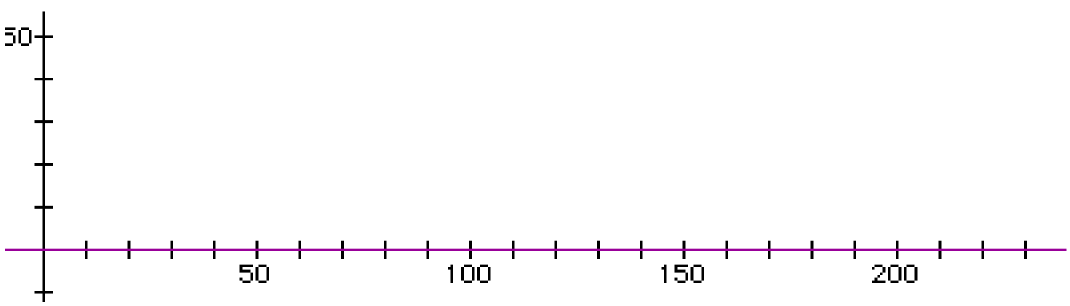
How might he improve the throw if it is bad?

Add your new values so that a 6 foot tall first baseman might catch the ball. Check it on your new graph.

II. A MORE REALISTIC MODEL.

Look at the graph drawn in part I. According to the graph, from what height did Cal throw the ball?

Suppose that the ball is thrown from a height of 4 feet. Change your equations to represent this difference. Sketch the graph below.



III. PRACTICE QUESTIONS.

Problems may be solved algebraically instead of graphically. Use both methods to try to use both methods to solve these problems. Draw and label graphs and show work.

1. A long jumper is running at a 25 ft/sec. If he takes off at a 37^0angle, how far will he travel before landing?

2. If he increase the angle to 42^0, how will this effect his jump?

3. A football place kicker is kicking a ball at a n angle of 50^0 and an initial velocity of 85 ft/s. What is the hang time of the punt?

4. What is the maximum height of the ball?

5. How far down the field will the ball travel?

6. Elmo the human cannonball is set to make his trip across the circus floor. The cannon is set to fire at 45 ft/s and at a 60^0 angle. How high must the catching net be set if it is sitting 46 feet away?

TEACHER PREPARATION GUIDE

NO ONE NEEDS A HALF-BAKED TEACHER

The objective of this lesson is to let students create the parametric equations used to describe the projectile flight path.

$$\mathbf{x} = \mathbf{v}_0\mathbf{t}\cos\alpha$$

$$\mathbf{y} = \mathbf{v}_0\mathbf{t}\sin\alpha - \frac{1}{2}\mathbf{g}\mathbf{t}^2$$

$\mathbf{v}_0$ = initial velocity α = angle of elevation

$$\mathbf{g} = 32\frac{\mathbf{ft}}{\mathbf{s}^2} \text{ or } 9.8\frac{\mathbf{m}}{\mathbf{s}^2}$$

PRIOR PREPARATION

Prepare students in the use of the calculator prior to the day of the lesson. **Stress** the use of lists, stat plots, and regression equations. It would be a good idea to **supply** some data that you generate from known linear and quadratic functions. The students can **practice** creating lists, graphing data plots, and determining regression equations. Students will probably need help in understanding the calculator display for each part.

Work on changing window settings. This exercise requires frequent changes in the window. A parabola might look linear if the window is not set correctly.

Graph some parametric functions to become familiar with this mode on the calculator. *To enter parametric mode, press* **Mode**, *and change* **Func** *to* **Par**.

Anticipate graphing difficulties from students. Most common problems will come from the window not being set correctly, lists not being erased, or the stat plot being set incorrectly. Check these things first.

The **window** is usually set with min values to 0 and setting max values for x, when $\alpha = 0^0$, and for y, when $\alpha = 90^0$.

PROJECTILE MOTION 1

Projectile Motion 1 has three parts. Part 1 has student determine the horizontal component. The equation used to generate the data was $\mathbf{y} = 100\mathbf{t}$. The velocity is 100 ft/s.

Emphasize to students that the graph does not represent the flight path of a ball but just the distance away versus time traveled.

Part II is used to examine the gravitational effect on a ball. The equation used to generate the data was $\mathbf{y} = 500 - .16\mathbf{t}^2$. The results were taken from the table generated by the calculator.

Part III examines the entire vertical component of the flight of a projectile. It starts at y=0 and ends at y=0. The equation used was $\mathbf{y} = 100\sin 90^0 - 16\mathbf{t}^2$.

For these exercises it is important that students see that the projectile has two components, the x distance and the y distance.

ANSWERS TO PRACTICE PROBLEMS

1. velocity is 100 ft/s

2. v= x/t

3. x=vt

4. x=85t

5. The free falling object has no initial velocity. The velocity obtained is due to the acceleration due to gravity.

6. $y=h-1/2\ gt^2$

7. The graph in part has an initial vertical velocity. It will be seen as a complete parabola.

8. The initial velocity of 100 ft/s.

PROJECTILE MOTION 2

Anticipate a lot of initial problems in graphing. The window should be set with xmax=80 and ymax=250. If the window is not set large enough, the parabola might appear linear.
The equations used to generate the curve were:

$$\mathbf{x} = 88\mathbf{t}\cos 30^0 \quad \mathbf{y} = 88\mathbf{t}\sin 30^0 - 16\mathbf{t}^2$$

Cal’s throw is not good. It is over 28 ft high when it passes first base.

He could either reduce the velocity of his throw or throw the ball at a smaller angle.

ANSWERS TO PRACTICE PROBLEMS.

1. The long jumper has a jump of 18.775 ft.

2. Increasing the angle would lengthen his flight .649 ft to 19.424 ft.

3. The hang time of the punt is 4.07 seconds.

4. The maximum height of the ball is 66.245 ft.

5. The ball will travel 265.013 ft, 88.338 yards (What a kick!)

6. Elmo’s catching net must be 12.81 ft high to catch him.

Performance Assessment

Teacher's Guide

Introduction

What do Tiger Wood's long drive, Evil Knievel, Jr.'s motor cycle jump over 23 limos, Randy Matson's winning shot-put toss, and Cal Ripken's throw to first base all have in common? They all deal with moving an object, a projectile, from one place to another in the most efficient way. Some of the criteria necessary for this efficiency are distance, accuracy, and time of flight. Cal Ripken needs our help in the analysis of his baseball throw to more effectively play defense at his new position, third base.

Objectives Covered

- Students will be able to draw graphs of projectile motions with sample motions of vertical, horizontal, and at other angles of elevation.
- Students will be able to write the parametric equations for both horizontal and vertical components of distance for projectile motions.
- Students will be able to determine the equations to describe projectile motion using the graphing calculator.
- Students will be able to describe how projectile motion is affected by changing the variables of initial velocity, the angle of elevation, and the acceleration of gravity.

Tools/Materials Needed for Assessment

- TI-82/83 graphing calculator
- Assessment materials included in this unit
- Scoring rubrics for Activity sheets #1 and #2 and the Extended Constructed Response and the Selected Response sections

Administering the Assessment

There are three assessment portions to this units. The Activity Sheets #1 and 2 have assessment rubrics which are included. The scoring key for the Extended Constructed Response is included with a scoring rubric for scoring guidance. The Selected Response answers are also included.

Performance Assessment

Student Response Sheet

I. Extended Constructed Response

A. The following data were collected for a 12-lb shot-put rolling horizontally in a straight line with a constant speed. Determine the relationship between the distance (x) and the time (t) in meters and seconds, respectively. Graph the data with your calculator and place the graph on the axis below and write the regression equation with proper variables on the equation line.

Equation: ______________________________

Time (seconds)	Distance (x) (meters)
0.00	0.0
0.40	4.8
1.00	12.0
1.60	19.2
2.10	25.2
2.60	31.2
3.30	39.6

Distance (m)

time (sec)

B. The same 12-lb shot-put was thrown vertically upward by a very strong wrestler, and the data were collected for height (y) versus time (t). Determine the relationship between the distance (y) and the time (t) in meters and seconds, respectively. Graph the data with your calculator and place the graph on the axis below and write the regression equation with proper variables on the equation line.

Equation:_________________________

Time (seconds)	Height (y) (meters)
0.00	0.00
0.40	4.02
0.80	6.46
1.20	7.34
1.60	6.66
2.00	4.40
2.40	0.58

Height (m)

time (sec)

C. Place the two equations into the calculator in parametric mode. Adjust the window with appropriate values (use angle of 45^{o} and initial velocity of 12 m/sec) in order to graph the complete motion. Draw the result of the graph below and show the window used.

Vertical Position (y) (m)

Window:

Horizontal Position (x) (m)

Questions: Complete the following questions using the data and results above.

1. Explain what the graph completed in part B represents for the shot-put. Include the maximum height and the range of the throw.

2. What would happen to the motion if the angle of elevation were increased significantly? Why?

3. What would happen to the motion if the initial velocity of the throw were increased? Why?

II. Selected Response Questions.

Directions: Complete the following questions and choose the best answer.

1. A bowling ball is moving with a constant velocity of 3.2 m/sec. In 2.5 sec the ball will travel
   (A) 1.28 m (B) 8.0 m (C) 0.78 m (D) 20 m

2. A basketball is thrown directly upward at 15 m/sec. If gravity is 9.80 m/sec^2, how far does the basketball rise in 2.2 sec?
   (A) 33 m (B) 11 m (C) 9.3 m (D) 57 m

3. All of the following are examples of projectile motion EXCEPT
   (A) a nine-iron golf shot onto the green
   (B) Michael Jordan's three-point jump shot
   (C) Mark McGwire's homer run into the right field stands
   (D) Tiger Wood's 40-foot birdie putt on the seventeenth hole

4. If the initial velocity of Tiger Wood's drive were lowered, this change would
   (A) increase in maximum height of his drive
   (B) reduce the distance of his drive
   (C) increase the distance of his drive
   (D) make no change in his drive

5. Suppose that Cal Ripken threw a baseball on the planet Mars where gravity is smaller than on the earth. Neglecting atmospheric resistance, the baseball would
   (A) travel the same path as on the earth.
   (B) go higher than on the earth.
   (C) go farther than on the earth.
   (D) go higher and travel farther than on the earth.

6. Suppose that Brady Anderson threw a baseball from right center field at an angle of elevation of 80^o, all of the following would occur EXCEPT
   (A) the distance thrown would be greater
   (B) the time of flight would be larger
   (C) the maximum height would be greater
   (D) the motion would be a parabola.

Use your graphing calculator to graph the following data for time 0 to 10. sec for a shot-put by an athlete trying out for the Olympic team. Use the result on problems 7 through 9.

$x = 8.5\,t$ and $y = 16t - 4.9t^2$

7. What is the range (distance) of the throw?
   (A) 13.0 (B) 28.0 (C) 13.6 (D) 38.1

8. What is the maximum height of the throw?
   (A) 13.0 (B) 28.0 (C) 13.6 (D) 38.1

9. Give the position (in ordered pairs) of the shot at 2.9 seconds.
   (A) (24.6, 5.19) (B) (5.19, 24.6) (C) (24.6, 32.2) (D) (24.6, 87.6)

Performance Assessment

Scoring Guide

I. Extended Constructed Response Exemplary Response

A. The data when graphed is a straight line because the velocity is constant. The linear regression equation is $x = 12t$.
B. The data when graphed is a parabola which is upside down. The times when the shot is on the ground are 0 and about 2.45 seconds. The regression equation is quadratic:

$$y = 12t - 4.9t^2$$

C. The graph of vertical position (y) versus horizontal position (x) using the parametric mode is a parabola. An acceptable window should include the entire parabola. The time should be from 0 to at least 10 seconds. The x values should vary from 0 to at least 30 meters with the y varying from 0 to at least 10 meters.

Questions:
1. A description of the graph should include the idea that the motion was parabolic with intercepts at $x = 0$ and $x = 2.45$ seconds. The maximum value of y where the shot stops rising was about 7.34 meters with the range of about 29.5 meters. The range represents the distance of the throw, the important variable in a shot-put event.
2. If the angle of elevation were increased significantly, the height and time of flight would increase. If the angle were increased to 45^{o}, then the range (distance) would increase. If the angle were to be increased beyond 45^{o}, then the distance would begin decreasing until 90^{o} was reached. At this point the range would be zero meters with all energy going into the vertical flight and time.
3. If the initial velocity were increased, the times, height, and distance (range) of the flight would be increased. The greater initial velocity would move the shot higher and farther in the same amount of time, plus the time for gravity to pull the projectile to the earth will be increased due to the greater energy input.

II. Selected Response Answers:
1. B 2. C 3. D 4. B 5. D 6. A 7. B 8. C 9. A

Scoring Rubric for Extended Constructed Response:

4 This response includes the following:
- all of the graphs are shown completely
- the two equation for parts A and B are accurate
- the graph and the window for part C were accurate
- the three questions contained valid reasoning

3 This response includes the following:
- all of the graphs are shown completely
- at least one of the equations is accurate and the others close
- the window and the graph for part C were reasonable
- at least two of the three questions are answered reasonably well

2 This response includes the following:
- the first two graphs are shown completely
- the equation for the linear <u>or</u> the quadratic data is accurate
- at least two of the three questions are answered reasonably well

1 This response includes the following:
- at least one graph, equation, and one question are accurate

0 No response